

Parametric Representation of Curves:

Suppose that a particle moves along a curve C in the xy -plane in such a way that its x - and y -coordinates, as functions of time, are

$$x = f(t), \quad y = g(t)$$

then we call these the parametric eqⁿs of motion for the particle and refer to C as the graph of the eqⁿ. The variable t is called the parameter for the eqⁿs.

Example: Find the graph of the parametric eqⁿs

$$x = \cos t, \quad y = \sin t \quad (0 \leq t \leq 2\pi)$$

Soln: One way to find the graph is to eliminate the parameter t by calculating

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1.$$

Thus the graph is contained in the unit circle $x^2 + y^2 = 1$.

Another way: Geometrically t can be interpreted as angle swept out by the radial line from origin to the point $(x, y) = (\cos t, \sin t)$.

Also $x = \text{radial distance from } (0,0) \text{ to } (x, y)$
 $= \sqrt{\cos^2 t + \sin^2 t} = 1$

on $t = 0, (x, y) = (1, 0)$

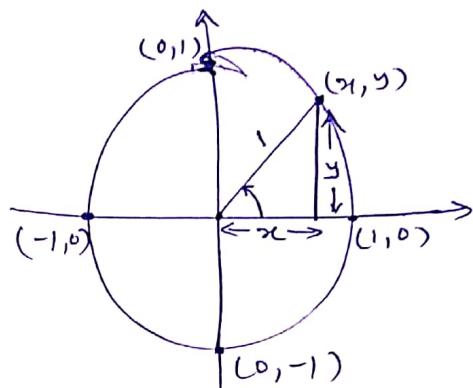
on $t = \frac{\pi}{2}, (x, y) = (0, 1)$

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$$\text{on } t = \frac{3\pi}{2}, (x, y) = (-1, 0)$$

$$\text{on } t = 2\pi, (x, y) = (1, 0)$$

\therefore As t increases from 0 to 2π , the point traces the circle counterclockwise.



Tracing of Parametric Curves:

Let the parametric eqns of the curve be $x = f(\theta)$, $y = g(\theta)$.

Then we draw/trace the graph of a parametric curve on the basis of the following points:

- (1) Symmetry: (a.) If $f(-\theta) = f(\theta)$, $g(-\theta) = -g(\theta)$, then the curve is symmetric w.r.t. x-axis.
- (b.) If $f(-\theta) = -f(\theta)$, $g(-\theta) = g(\theta)$, then the curve is symmetric w.r.t. y-axis.
- (c) If $f(-\theta) = -f(\theta)$, $g(-\theta) = -g(\theta)$, then the curve is symmetric in opposite quadrant.
- (2) Origin: Put $x=0$ and find θ . If for this value of θ y is also equal to zero then the curve passes through origin.

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(3) Asymptotes: find the asymptotes of the curve if any.

(4) Points: Find the intersection points on the axes.

Find the points where $\frac{dy}{dx} = 0$ or ∞ .

find the increasing / decreasing nature of the curve
from the sign of $\frac{dy}{dx}$ for different values of t .

(5) Region: find the region where the curve is situated.

Example:

Trace the following curve:
 $x = a \cos \theta$, $y = a \sin \theta$

$$x = a(\theta + \sin \theta), y = a \frac{(1 - \cos \theta)}{\sin \theta}$$

Solution:

(1) Symmetry: as $f(-\theta) = -f(\theta)$ & $g(-\theta) = g(\theta)$.

\therefore Curve is symmetric w.r.t. y -axis.

(2) Origin: On putting $x=0 \Rightarrow \theta=0$

and at $\theta=0$, $y=0$.

\therefore Curve passes through origin.

(3) Points: $\frac{dx}{d\theta} = a(1 + \cos \theta)$, $\frac{dy}{d\theta} = a \sin \theta \Rightarrow \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$

$$\therefore \frac{dy}{dx} = \frac{a \sin \theta}{a(1 + \cos \theta)} = \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{1 + 2 \cos^2 \frac{\theta}{2} - 1} = \tan \frac{\theta}{2}$$

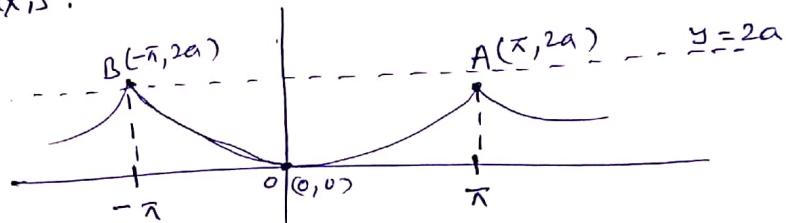
θ	0	$\pi/2$	π	$-\pi/2$	$-\pi$
x	0	$a(1 + \frac{\pi}{2})$	$a\pi$	$-a(1 + \frac{\pi}{2})$	$-a\pi$
y	0	a	$2a$	a	$2a$
$\frac{dy}{dx}$	0	1	∞	-1	$-\infty$

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From the first column of the table, it is clear that curve passes through origine $(0,0)$ and the slope of the tangent at origine is zero that is x -axis is the tangent line of the curve at origine.

From the third column i.e. at the point $(a\pi, 2a)$ it is clear that at this point tangent line is y -axis parallel to y -axis.

Maximum value of y is $2a$ and minimum is 0 .



sketching the path of a parametric curve by eliminating the parameter:

Example: Describe the path $x = \sin \pi t$, $y = \cos 2\pi t$ for $0 \leq t \leq 0.5$

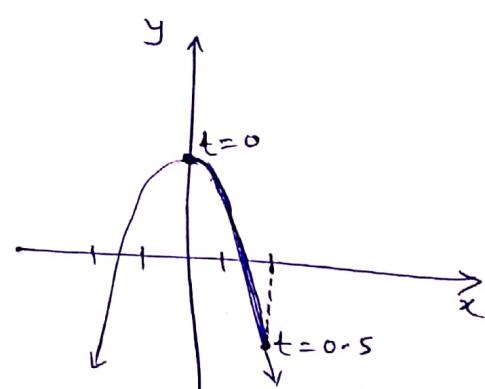
Soln: $\therefore \cos 2\pi t = 1 - 2\sin^2 \pi t$
 $\therefore y = 1 - x^2$

which is a cartesian eqⁿ of a parabola.

For critical number $\frac{dy}{dx} = -4x = 0 \Rightarrow x = 0$
 $\Rightarrow y = 1$

which locate the vertex of the parabola at $(0,1)$.

The curve is oriented as dashed curve involves only part of the right side of the parabola from the pt. $(0,1)$ when $t=0$ & to the pt. $(1,-1)$ when $t=\frac{1}{2}$.



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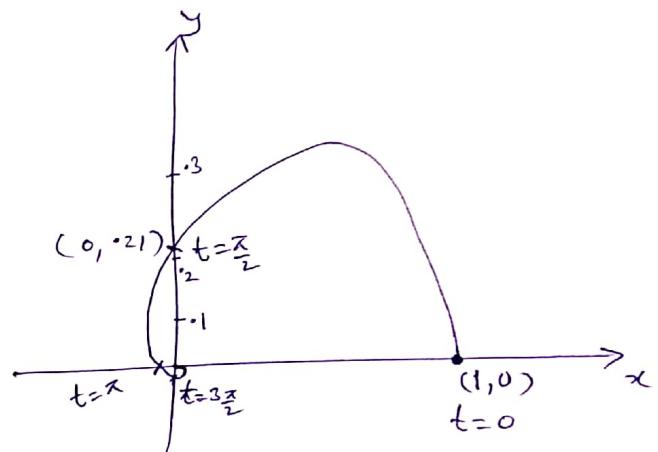
Example: Discuss the path of the curve described by the parametric eqn.

$$x = e^{-t} \cos t, y = e^{-t} \sin t \text{ for } t \geq 0.$$

Soln. $\therefore \sqrt{x^2 + y^2} = e^{-t}$

Because e^{-t} decreases as t increases, it follows that $P(x, y)$ gets closer and closer to the origin as t increases.

t	x	y
0	1	0
$\frac{\pi}{2}$	0	$e^{-\frac{\pi}{2}} = (0.21)$
π	$-e^{-\pi} = (-0.04)$	0
$\frac{3\pi}{2}$	0	$-e^{-\frac{3\pi}{2}} = (-0.01)$
2π	0	0



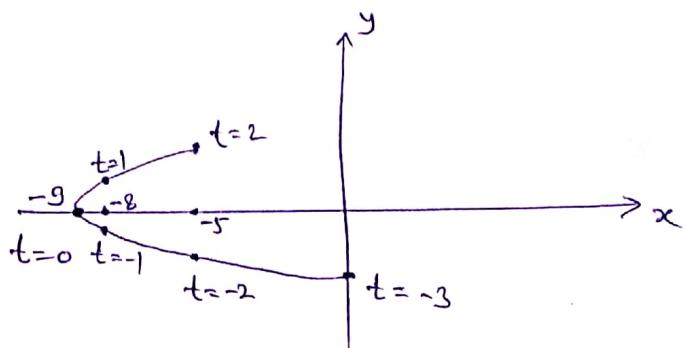
As we have no convenient way of eliminating the parameter so we write out a table of values (x, y) .

Example: Sketch the path of the curve

$$x = t^2 - 9, y = \frac{1}{3}t \text{ for } -3 \leq t \leq 2.$$

Soln.

t	x	y
-3	0	-1
-2	-5	$-\frac{2}{3}$
-1	-8	$-\frac{1}{3}$
0	-9	0
1	-8	$\frac{1}{3}$
2	-5	$\frac{2}{3}$



Otherwise, we can eliminate the parameter to obtain
a cartesian eq?

$$x = (3y)^2 - 9 = 9(y^2 - 1).$$