Short methods of finding the particular integrals :-

(a) If $\boxed{F(x) = e^{ax}}$ or P.I. corresponding to a term of form e^{ax}.

then eqn is

$$f(D)y = e^{ax}$$

and

$$P.I. = \frac{1}{f(D)} e^{ax}$$

then

$$\boxed{P.I. = \frac{1}{f(a)} e^{ax}}, \text{ provided } f(a) \neq 0.$$

Suppose that $f(a) = 0$, or 'a' is a root of $f(D)$.
then, suppose that $f(D) = (D-a)\phi(D)$, s.t. $\phi(a) \neq 0$.

then

$$\frac{1}{f(D)} e^{ax} = \frac{1}{(D-a)\phi(D)} e^{ax}$$

$$= \frac{1}{(D-a)} \frac{1}{\phi(a)} e^{ax}$$

$$= \frac{1}{\phi(a)} x e^{ax}$$

If a is a root of $f(D)$, r times, then

$$f(D) = (D-a)^r \phi(D)$$

$$\frac{1}{f(D)} e^{ax} = \frac{1}{(D-a)^r \phi(D)} e^{ax} = \frac{1}{\phi(a)} \times \frac{x^r e^{ax}}{(r)!}$$

$$\boxed{\frac{1}{f(D)} e^{ax} = \frac{x^{r} e^{ax}}{\lfloor r \ \phi(a)}}$$

Example :- $\frac{d^3y}{dx^3} + y = 3 + e^{-x} + 5e^{2x}$ —— (1)

[try to find y_c complementary function yourself],

The P.I. of (1) is

$$y_p = \frac{1}{(D^3+1)} (3 + e^{-x} + 5e^{2x})$$

$$y_p = \frac{1}{(D+1)(D^2-D+1)} (3 + e^{-x} + 5e^{2x}) \quad \text{—— (2)}$$

P.I. Corresponding to 3 is

$$= \frac{1}{(D+1)(D^2-D+1)} (3 \cdot e^{0 \cdot x})$$

$$= 3 \times \frac{1}{0^3+1} = 3$$

P.I. Corresponding to $5e^{2x}$

$$= \frac{1}{D^3+1} 5e^{2x} = \frac{5e^{2x}}{2^3+1}$$

$$= \frac{5}{9} e^{2x}.$$

P.I. Corresponding to e^{-x}

$$= \frac{1}{D^3+1} e^{-x}$$

[$D^3+1=0$ as $D=-1$]

$$= \frac{1}{(D+1)(D^2-D+1)} e^{-x}$$

$$= \frac{1}{(D+1)} \frac{e^{-x}}{\{(-1)^2+1+1\}} = \frac{1}{(D+1)} \cdot \frac{e^{-x}}{3}$$

$$= \frac{1}{3} \cdot x e^{-x} \qquad \left\{ \text{as } \frac{1}{(D-a)} e^{ax} = x e^{ax} \right\}$$

therefore

$$y_p = 3 + \frac{1}{3} x e^{-x} + \frac{5}{9} e^{2x}$$

general soln of ①

$$y = y_c + y_p$$

$$\boxed{y = y_c + 3 + \frac{1}{3} x e^{-x} + \frac{5}{9} e^{2x}}$$

Exercise :- (1) Solve $\frac{d^3y}{dx^3} - 1 = (e^x + 1)^2$

Exercise (2): Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 3e^{\frac{5}{2}x}$,

(b) If $F(x) = x^m$, m being a positive integer;-

Example:- Solve $(D^3+3D^2+2D)y = x^2$ —— (I)

Complementary function y_c of (I) is $C_1 + C_2e^{-2x} + C_3e^{-x}$.

(verify it)

The P.I. is

$$y_p = \frac{1}{D^3+3D^2+2D}\, x^2$$

$$= \frac{1}{2D+3D^2+D^3}\, x^2$$

$$= \frac{1}{2D\left[1+\frac{3}{2}D+\frac{D^2}{2}\right]}\, x^2$$

$$= \frac{1}{2D}\left[1+\frac{3}{2}D+\frac{D^2}{2}\right]^{-1} x^2$$

$$= \frac{1}{2D}\left[1-\frac{3}{2}D-\frac{D^2}{2}+\frac{9}{4}D^2+\frac{3}{2}D^3+\frac{D^4}{4}+\dots\right] x^2$$

$$= \frac{1}{2D}\left[x^2-\frac{3}{2}\times 2x-\frac{2}{2}+\frac{9}{4}\times 2+0+0\dots\right]$$

$$= \frac{1}{2D}\left[x^2-3x-1+\frac{9}{2}\right] = \frac{1}{2D}\left[x^2-3x+\frac{7}{2}\right]$$

$$= \frac{1}{2}\left[\frac{x^3}{3}-\frac{3}{2}x^2+\frac{7}{2}x\right] \qquad \left\{\frac{1}{D} = D^{-1} = \text{Integration}\right\}$$

$$y_p = \frac{x}{12}\left[2x^2-9x+21\right] \qquad \left\{\frac{1}{D}x = \int x\,dx\right\}$$

then soln of (1) is

$$y = y_c + y_p$$

$$= c_1 + c_2 e^{-2x} + c_3 e^{-x} + \frac{x}{12}(2x^2 - 9x + 21).$$

Hence, when $\frac{1}{f(D)} x^m$ is to be evaluated, raise $f(D)$ to the ~~pow~~ (-1)th power, arranging the terms in ascending powers of D. Then write expansion of $[f(D)]^{-1}$ and operate to x^m, the result will be P.I. corresponding to x^m.

Exercise:- Solve $\frac{d^3y}{dx^3} + 8y = x^4 + 2x + 1$

(c) If $f(x) = \sin ax$ or $f(x) = \cos ax$

then $\frac{1}{\phi(D^2)} \sin ax = \frac{1}{\phi(-a^2)} \sin ax$ $[\phi(-a^2) \neq 0]$

and $\frac{1}{\phi(D^2)} \cos ax = \frac{1}{\phi(-a^2)} \cos ax$

[Provided $\phi(-a^2) \neq 0$].

<u>Example</u> :- $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - \frac{dy}{dx} - y = \cos 2x$

or

$(D^3 + D^2 - D - 1)y = \cos 2x$

~~To~~ Complementary function

$y_c = c_1 e^x + (c_2 + x c_3) e^{-x}$

and Particular Integral

$$= \frac{1}{D^3 + D^2 - D - 1} \cos 2x$$

$$= \frac{1}{D^2(D+1) - 1(D+1)} \cos 2x$$

$$= \frac{1}{(D+1)} \left[\frac{1}{D^2 - 1} \cos 2x \right]$$

$$= \frac{1}{(D+1)} \left[\frac{1}{-4-1} \cos 2x \right]$$ as $\left[\frac{1}{\phi(D^2)} \cos ax = \frac{1}{\phi(-a^2)} \cos ax \right]$

$$= \frac{1}{(D+1)} \left[-\frac{1}{5} \cos 2x \right]$$

$$y_p = -\frac{1}{5}\left[\frac{1}{(D+1)}\cos 2x\right]$$

$$= -\frac{1}{5}\left[\frac{(D-1)}{(D+1)(D-1)}\cos 2x\right]$$

$$= -\frac{1}{5}(D-1)\left[\frac{1}{D^2-1}\cos 2x\right]$$

$$= -\frac{1}{5}(D-1)\left[\frac{1}{-4-1}\cos 2x\right]$$

$$= -\frac{1}{5}(D-1)\left[-\frac{1}{5}\cos 2x\right]$$

$$= \frac{1}{25}(D-1)\cos 2x$$

$$= \frac{1}{25}\left[D(\cos 2x) - \cos 2x\right]$$

$$= \frac{1}{25}\left[-2\sin 2x - \cos 2x\right]$$

$$y_p = -\frac{2}{25}\sin 2x - \frac{\cos 2x}{25}$$

Hence general sol^n of (1) is given as

$$y = y_c + y_p$$

$$\boxed{y = c_1 e^{x} + (c_2 + x c_3)e^{-x} - \frac{2}{25}\sin 2x - \frac{\cos 2x}{25}}$$

Also remember following formulas when $\phi(-a^2)$ is zero,

$$\frac{1}{D^2+a^2}\cos ax = \frac{x\sin ax}{2a}$$

and

$$\frac{1}{D^2+a^2}\sin ax = -\frac{x\cos ax}{2a}$$

Example :- $\frac{d^2y}{dx^2} - y = \cos x$

Here $y_c = c_1 e^x + c_2 e^{-x}$

and

P.I. $y_p = \frac{1}{D^2-1}\cos x$ [and $-1^2-1=0$]

then

$$y_p = \frac{x\sin x}{2\times 1}$$

$$\boxed{y_p = \frac{x}{2}\sin x}$$

Then, general sol^n of (1) is given as

$$\boxed{y = c_1 e^x + c_2 e^{-x} + \frac{a}{2}\sin x}$$

Exercise :- Solve $\frac{d^2y}{dx^2} - 4y = 2\sin\frac{x}{2}$.

(d) If $F(x) = e^{ax} V$, where V is any function of x.

Then particular Integral of equation

$$f(D) y = e^{ax} V$$

is

$$P.I. = \frac{1}{f(D)} e^{ax} V$$

$$= e^{ax} \frac{1}{f(D+a)} V$$

Example :-

$$\frac{d^2y}{dx^2} + y = x e^{2x} \quad \text{--- (I)}$$

~~Complete fund~~

Complementary function of (I) is

$$\boxed{y_c = c_1 \sin x + c_2 \cos x}$$

and P.I. Corresponding to $x e^{2x}$ is

$$y_p = \frac{1}{D^2+1} x e^{2x}$$

$$= e^{2x} \left(\frac{1}{(D+2)^2+1} x \right)$$

$$= e^{2x} \frac{1}{D^2+4D+5} x$$

$$y_p = e^{2x} \frac{1}{5\left[1+\frac{4}{5}D+\frac{D^3}{5}\right]} \cdot x$$

$$= e^{2x} \cdot \frac{1}{5}\left[1+\frac{4}{5}D+\frac{D^3}{5}\right]^{-1} x$$

$$= \frac{1}{5}e^{2x} \cdot \left[1-\frac{4}{5}D-\frac{D^3}{5}+\cdots\right]x$$

$$= \frac{1}{5}e^{2x}\left[x-\frac{4}{5}-0\right]$$

$$\boxed{y_p = \frac{1}{5}xe^{2x} - \frac{4}{25}e^{2x}}$$

Hence general soln of given eqn is

$$\boxed{y = c_1 \sin x + c_2 \cos x + \frac{1}{5}xe^{2x} - \frac{4}{25}e^{2x}}$$

Exercise: (1) Solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{2x}\sin x$

(2) Solve $\frac{d^2y}{dx^2} + 2y = x^2e^{3x} + e^x\cos 2x$

(e) If $F(x) = xV$, when V is a function of 'x'.

then P.I. Corresponding to $f(D)y = xV$ is

$$\boxed{y_p = \frac{1}{f(D)} xV = \left\{ x - \frac{1}{f(D)} f'(D) \right\} \frac{1}{f(D)} V}$$

Example:- $\frac{d^2y}{dx^2} + y = x e^{2x}$

P.I. of given eqn

$$y_p = \frac{1}{(D^2+1)} x e^{2x}$$

$$= \left\{ x - \frac{1}{D^2+1} . 2D \right\} \left(\frac{1}{D^2+1} e^{2x} \right)$$

$$= \left\{ x - \frac{2D}{D^2+1} \right\} \left[\frac{1}{4+1} e^{2x} \right]$$

$$= \left\{ x - \frac{2D}{D^2+1} \right\} \left\{ \frac{1}{5} e^{2x} \right\}$$

$$= \frac{1}{5} x e^{2x} - \frac{2}{5} D . \left[\frac{1}{D^2+1} e^{2x} \right]$$

$$= \frac{1}{5} x e^{2x} - \frac{2}{5} D \left\{ \frac{1}{5} e^{2x} \right\}$$

$$= \frac{1}{5} x e^{2x} - \frac{2}{25} D e^{2x} = \frac{1}{5} x e^{2x} - \frac{2 \times 2}{25} e^{2x}$$

$$\boxed{y_p = \frac{1}{5} x e^{2x} - \frac{4}{25} e^{2x}}$$

3. $\dfrac{d^3y}{dx^3} + 2\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = e^{2x} + x^2 + x.$

4. $\dfrac{d^2y}{dx^2} + 4y = \sin 3x + e^x + x^2.$

5. $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = x + e^{mx}.$

6. $(D^2 - a^2)y = e^{ax} + e^{nx}.$

7. $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 8y = x.$

8. $\dfrac{d^4y}{dx^4} + \dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} = x^2(a + bx).$

9. $\dfrac{d^3y}{dx^3} - 13\dfrac{dy}{dx} + 12y = x.$

10. $\dfrac{d^4y}{dx^4} + 2n^2\dfrac{d^2y}{dx^2} + n^4y = \cos mx.$

11. $\dfrac{d^4y}{dx^4} + 2\dfrac{d^2y}{dx^2} + y = x^2 \cos x.$

12. $\dfrac{d^2y}{dx^2} + a^2y = \sec ax.$

13. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = x^2e^{3x}.$

14. $(D^2 + n^2)y = e^x x^4.$

15. $\dfrac{d^4y}{dx^4} - a^4y = x^4.$

16. $\dfrac{d^4y}{dx^4} - 2\dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} = x.$

17. $\dfrac{d^4y}{dx^4} - y = e^x \cos x.$

18. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = \sin 2x.$

19. $\dfrac{d^3y}{dx^3} - 7\dfrac{dy}{dx} - 6y = e^{2x}(1 + x).$

20. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 4y = e^x \cos x.$

21. $\dfrac{d^3y}{dx^3} + 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + y = e^{-x}.$

22. $(D^4 - 2D^3 - 3D^2 + 4D + 4)y = x^2e^x.$

23. $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - y = xe^x + e^x.$

24. $\dfrac{d^2y}{dx^2} - y = x \sin x + (1 + x^2)e^x.$

25. $(D^2 - 4D + 3)y = e^x \cos 2x + \cos 3x.$

26. $(D^3 - 3D^2 + 4D - 2)y = e^x + \cos x.$

27. $(D^2 - 9D + 20)y = 20x.$

28. $(D^3 - 3D^2 + 4)y = e^{3x}.$

29. $\dfrac{d^3y}{dx^3} + y = e^{2x} \sin x + e^{\frac{x}{2}} \sin \dfrac{x\sqrt{3}}{2}.$

... constants, is equal