

Volumes by Slicing, Disks and Washers:

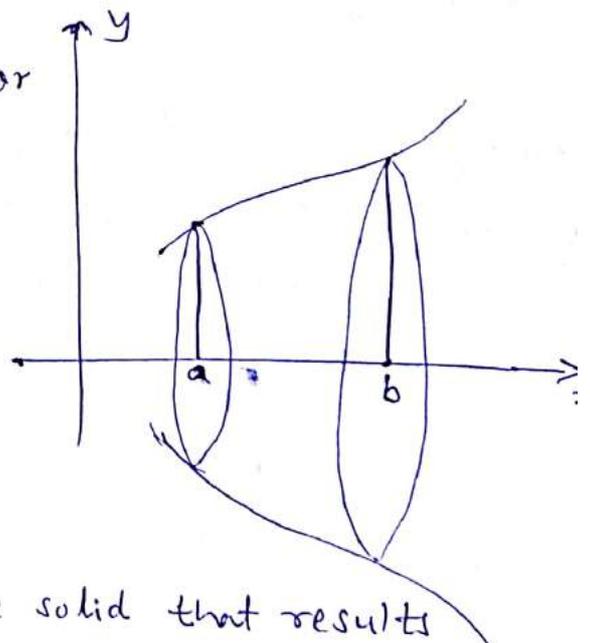
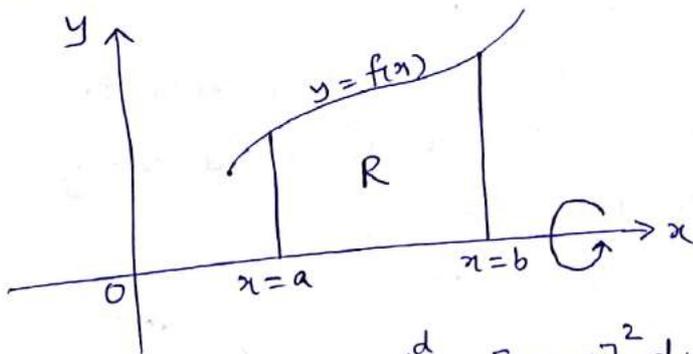
Method of Disks:

Let f be continuous and nonnegative on $[a, b]$, and let R be the region that is bounded by the lines $x=a$ and $x=b$, the x -axis and by the curve $y=f(x)$. Then the volume of the solid of revolution that is generated by revolving the region R about the x -axis is

$$= \int_a^b \pi y^2 dx$$

or
$$V = \int_a^b \pi [f(x)]^2 dx$$

Note: Here cross sections are perpendicular to the axes.



Note:
$$V = \int_c^d \pi [u(y)]^2 dy$$

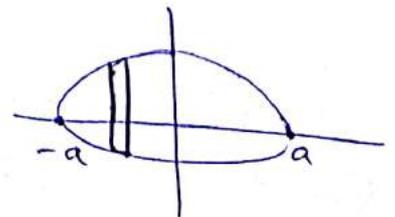
Question 21. Find the volume of the solid that results when the region above the x -axis and below the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ($a > 0, b > 0$) is

revolved about the x -axis.

Soln.

$$V = \int_{-a}^a \pi y^2 dx = \int_{-a}^a \pi b^2 \left(1 - \frac{x^2}{a^2}\right) dx$$

$$= 2 \int_0^a \pi b^2 \left(1 - \frac{x^2}{a^2}\right) dx = 2\pi b^2 \left(x - \frac{x^3}{3a^2}\right) \Big|_0^a = \frac{4}{3} \pi a b^2$$



Method of Washers:

Let f and g be continuous and nonnegative on $[a, b]$ and suppose that $f(x) \geq g(x)$ for all x in $[a, b]$. Let R be the region that is bounded by the lines $x = a$ and $x = b$ and bounded above by the curve $y = f(x)$, bounded below by the curve $y = g(x)$. Then the volume of the solid of revolution generated by revolving the region R about the x -axis is

given by

$$V = \int_a^b \pi ([f(x)]^2 - [g(x)]^2) dx$$

Note: $V = \int_c^d \pi [\{w(y)\}^2 - \{v(y)\}^2] dy$

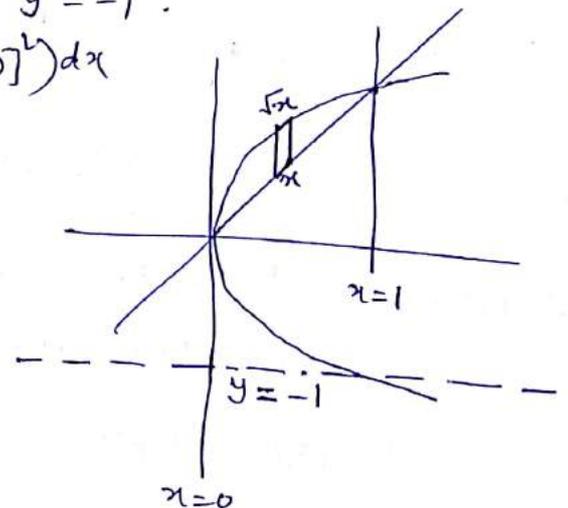
Question 27. Find the volume of the solid that results when the region enclosed by $x = y^2$ and $x = y$ is revolved about the line $y = -1$.

$$V = \pi \int_a^b ([f(x)]^2 - [g(x)]^2) dx$$

Solⁿ:

$$V = \pi \int_0^1 [(1 + \sqrt{x})^2 - (1 + x)^2] dx$$

$$= \frac{\pi}{2}$$

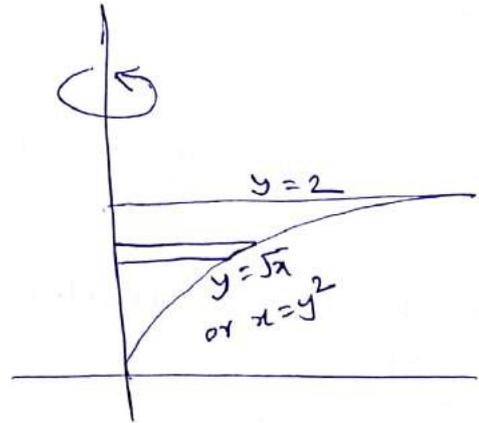


$\therefore f(x)$ is the distance from the axis of revolution.
Similarly $g(x)$ is the distance from axis of revolution.

Example. Find the volume of the solid generated ^(P-409)
when the region enclosed by $y = \sqrt{x}$, $y = 2$ and
 $x = 0$ is revolved about the y -axis.

Soln.

$$\begin{aligned} V &= \int_c^d \pi [u(y)]^2 dy \\ &= \pi \int_0^2 \pi (y^2)^2 dy \\ &= \frac{32\pi}{5} \end{aligned}$$

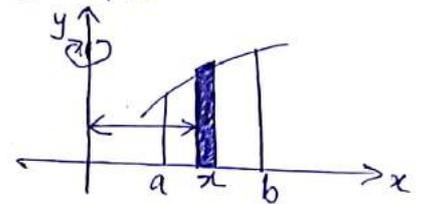


Volumes by Cylindrical Shells:

(P-413/414)

(1.) About the y-axis: Let f be cts. and nonnegative on $[a, b]$ and Let R be the region that is bounded above by $y = f(x)$, the x -axis and by the lines $x = a$ and $x = b$. Then the volume V of the solid of revolution generated by revolving the region R about the y -axis is

$$V = \int_a^b 2\pi x f(x) dx$$



Note: The cross section of R is taken parallel to the axis of revolution.

Similarly, we can find the volume by cylindrical shells about the x -axis which is $V = \int_c^d 2\pi y g(y) dy$

(2.) Let f and g be continuous and non-negative on $[a, b]$ and suppose that $f(x) \geq g(x)$ for all x in $[a, b]$. Let R be the region that is bounded by the lines $x = a$ and $x = b$, and bounded above by the curve $y = f(x)$, bounded below by the curve $y = g(x)$. Then the volume of the solid of revolution generated by revolving the region R about the y -axis is given by

$$V = \int_a^b 2\pi x [f(x) - g(x)] dx$$

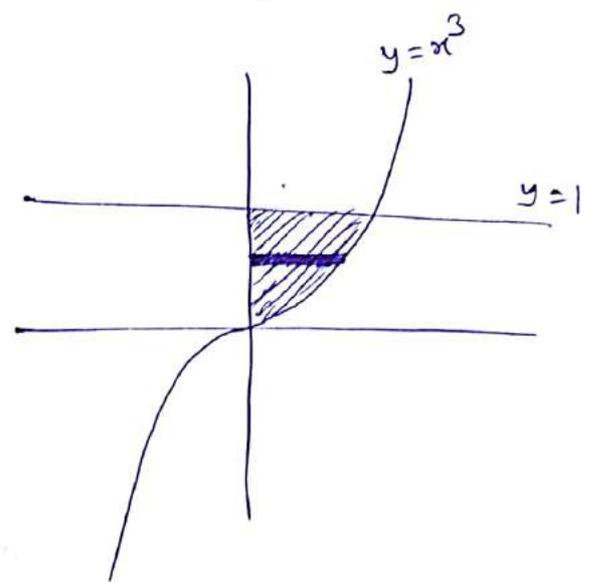
Similarly for x -axis is given by $V = \int_c^d 2\pi y [u(y) - v(y)] dy$

...olution:

(n

Question 19. Use cylindrical shells to find the volume of the solid that is generated when the region that is enclosed by $y = x^3$, $y = 1$, $x = 0$ is revolved about the line $y = 1$.

Solution:



$$V = \int_0^1 2\pi(1-y) g(y) dy$$

$$= 2\pi \int_0^1 (1-y) y^{\frac{1}{3}} dy$$

$$= 2\pi \left(\frac{3}{4} y^{\frac{4}{3}} - \frac{3}{7} y^{\frac{7}{3}} \right) \Big|_0^1 = 2\pi \left(\frac{3}{4} - \frac{3}{7} \right) = \frac{18\pi}{28} = \frac{9\pi}{14}$$

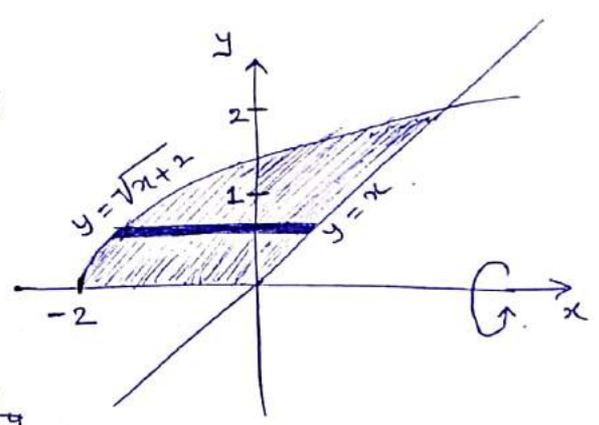
Q.4. $V = \int_c^d 2\pi y [u(y) - v(y)] dy$

$$V = \int_0^2 2\pi y [y - (y^2 - 2)] dy$$

$$= \int_0^2 2\pi y (\sqrt{x+2} - x) [(y^2 - 2) - y] dy$$

$$= 2\pi \left[\frac{y^4}{4} - \frac{y^2}{2} - \frac{y^3}{3} \right]_0^2 = 2\pi \left(4 - 4 - \frac{8}{3} \right)$$

$$= \frac{16}{3} \pi$$



Length of a Plane Curve or Arc Length:

(P-417-419)

(1.) If $y = f(x)$ is a smooth (f' be continuous) curve on the interval $[a, b]$ then the arc length L of this curve over $[a, b]$ is given by

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} \, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

(2.) Similarly, if $x = g(y)$ is a smooth curve on $[c, d]$,

Then arc length

$$L = \int_c^d \sqrt{1 + [g'(y)]^2} \, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$$

(3.) Arc Length formula for Parametric Curve:

If the parametric eq^{ns} are $x = x(t)$, $y = y(t)$;

$a \leq t \leq b$ and if $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are continuous fun^s.

for $a \leq t \leq b$, then

$$\text{arc length } L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

(4.) Arc Length in Polar Coordinates:

The length of a polar curve $r = f(\theta)$ for $\alpha \leq \theta \leq \beta$ is

$$\text{given by } s = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

(4)

Question. 11 Find arc length

$$x = \cos 2t, \quad y = \sin 2t \quad 0 \leq t \leq \frac{\pi}{2}$$

Soln

$$L = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$= \int_0^{\pi/2} \sqrt{(-2\sin 2t)^2 + (2\cos 2t)^2} dt$$

$$= 2 [t]_0^{\pi/2} = \pi$$

Example 1. Find the arc length of the curve $y = x^{3/2}$ from $(1, 1)$ to $(2, 2\sqrt{2})$.

Question.

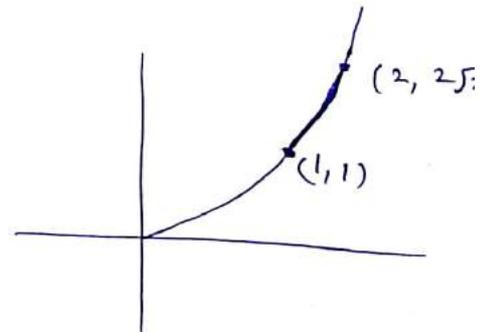
Soln

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

$$= \int_1^2 \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2} dx$$

$$= \int_1^2 \sqrt{1 + \frac{9}{4}x} dx$$

$$= \frac{8}{27} \left[\left(\frac{22}{4}\right)^{3/2} - \left(\frac{13}{4}\right)^{3/2} \right]$$



Area of a surface of revolution:

(P-423)

(1.) If f is a smooth, nonnegative function on $[a, b]$ then the surface area S generated by revolving the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ about the x -axis is

$$S = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} dx$$

$$\text{or } S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

(2.) If g is a smooth, nonnegative curve on $[c, d]$, then the area of the surface that is generated by revolving the portion of a curve $x = g(y)$ between $y = c$ and $y = d$ about the y -axis is

$$S = \int_{y=c}^{y=d} 2\pi g(y) \sqrt{1 + [g'(y)]^2} dy$$

$$\text{or } S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

(3.) The area of the surface generated by revolving the curve $x = x(t)$, $y = y(t)$; $a \leq t \leq b$, if x & y are smooth, about x -axis

$$S = \int_a^b 2\pi y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Similarly, about y -axis

$$S = \int_a^b 2\pi x(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Question-23 By revolving the semicircle $x = r \cos t$, $y = r \sin t$; $0 \leq t \leq \pi$, about the x -axis, show that the surface area of a sphere of radius r is $4\pi r^2$.

Solⁿ

$$\begin{aligned} S &= \int_0^{\pi} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_0^{\pi} 2\pi r \sin t \sqrt{(-r \sin t)^2 + (r \cos t)^2} dt \\ &= 2\pi r^2 \int_0^{\pi} \sin t dt = 2\pi r^2 (-\cos t)_0^{\pi} \\ &= 4\pi r^2 \end{aligned}$$

□

Example(2). Find the area of the surface that is generated by revolving the portion of the curve $y = x^2$ between $x=1$ and $x=2$ about the y-axis.

Solⁿ.

$$S = \int_{y=c}^{y=d} 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

$$= \int_1^4 2\pi \sqrt{y} \sqrt{1 + \left(\frac{1}{2\sqrt{y}}\right)^2} dy$$

$$= \pi \int_1^4 \sqrt{4y+1} dy = \frac{\pi}{6} (17^{3/2} - 5^{3/2})$$

□

Reduction Formulas:

$$(1.) \int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

Proof:

$$\text{let } I_n = \int \cos^n x \, dx$$

$$= \int \frac{\cos^{n-1} x}{\text{I}} \frac{\cos x \, dx}{\text{II}}$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \cdot \sin x \cdot \sin x \, dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int (1 - \cos^2 x) \cos^{n-2} x \, dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \underbrace{\int \cos^n x \, dx}_{I_n}$$

$$n I_n = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx$$

$$\therefore \int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

Question: Evaluate $\int \cos^4 x \, dx$.

(2.) Similarly,

$$\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

$$(3.) \int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx$$

Proof:

$$\begin{aligned} \text{Let } I_n &= \int \tan^n x \, dx \\ &= \int \tan^{n-2} x \cdot \tan^2 x \, dx \\ &= \int \tan^{n-2} x (\sec^2 x - 1) \, dx \\ &= \int \tan^{n-2} x \sec^2 x \, dx - \int \tan^{n-2} x \, dx \\ I_n &= \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx \end{aligned}$$

$$(4.) \int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx$$

Proof:

$$\begin{aligned} \text{Let } I_n &= \int \sec^n x \, dx \\ &= \int \sec^{n-2} x \cdot \sec^2 x \, dx \\ &= \sec^{n-2} x \cdot \tan x - \int (n-2) \sec^{n-3} x \cdot \sec x \tan x \cdot \tan x \, dx \\ &= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \tan^2 x \, dx \\ &= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) \, dx \\ I_n &= \sec^{n-2} x \tan x - (n-2) I_n + (n-2) \int \sec^{n-2} x \, dx \\ (n-1) I_n &= \sec^{n-2} x \tan x + (n-2) \int \sec^{n-2} x \, dx \\ I_n &= \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx \end{aligned}$$

Example 1 Find the reduction formula for $\int \sin^n x \, dx$,
 n being a positive integer and hence evaluate
 $\int_0^{\pi/2} \sin^n x \, dx$.

Solⁿ $\therefore \int \sin^n x \, dx = -\frac{\cos x \sin^{n-1} x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx$

$$\begin{aligned} \therefore \int_0^{\pi/2} \sin^n x \, dx &= \left. -\frac{\cos x \sin^{n-1} x}{n} \right|_0^{\pi/2} + \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x \, dx \\ &= 0 + \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x \, dx \quad \text{--- (1)} \end{aligned}$$

Let $I_n = \int_0^{\pi/2} \sin^n x \, dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x \, dx$ [from (1)]

or $I_n = \frac{n-1}{n} I_{n-2}$ --- (2)

Replacing n by $n-2, n-4, \dots, 3, 2$ in (2), we obtain

$$I_{n-2} = \frac{n-3}{n-2} I_{n-4}$$

$$I_{n-4} = \frac{n-5}{n-4} I_{n-6}$$

.....

$$I_3 = \frac{2}{3} I_1 = \frac{2}{3}$$

$$I_2 = \frac{1}{2} I_0 = \frac{1}{2} \cdot \frac{\pi}{2}$$

and $I_1 = \int_0^{\pi/2} \sin x \, dx = -\cos x \Big|_0^{\pi/2} = 1$

$$I_0 = \int_0^{\pi/2} 1 \cdot dx = \frac{\pi}{2}$$

P.T.O. (8)

$$\int_0^{\frac{\pi}{2}} \sin^n x \, dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot \dots \cdot \frac{2}{3}, \text{ if } n \text{ is odd}$$

$$\frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot \dots \cdot \frac{1}{2} \cdot \frac{\pi}{2}, \text{ if } n \text{ is even}$$

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot \dots \cdot \frac{2}{3}$$

$$\frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot \dots \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

□

Example 2. Obtain the reduction formula for $\int \sin^m x \cos^n x dx$,
 m and n being positive integers.

Evaluate $\int_0^{\pi/2} \sin^m x \cos^n x dx$.

Solⁿ: $\therefore \int \sin^m x \cos^n x dx = \int \underbrace{\sin^{m-1} x}_I (\underbrace{\sin x \cos^n x}_II) dx$

$$= \sin^{m-1} x \left(-\frac{\cos^{n+1} x}{n+1} \right) + \frac{m-1}{n+1} \int \sin^{m-2} x \cos^{n+2} x dx$$

$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1} \int \sin^{m-2} x \cos^n x \cos^2 x dx$$

$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1} \int \sin^{m-2} x \cos^n x (1 - \sin^2 x) dx$$

$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1} \int \sin^{m-2} x \cos^n x dx$$

$$- \frac{m-1}{n+1} \int \sin^m x \cos^n x dx$$

$$\left(1 + \frac{m-1}{n+1}\right) \int \sin^m x \cos^n x dx = -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1} \int \sin^{m-2} x \cos^n x dx$$

$$\therefore \int \sin^m x \cos^n x dx = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} \int \sin^{m-2} x \cos^n x dx$$

Now $\int_0^{\pi/2} \sin^m x \cos^n x dx = -\left[\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} \right]_0^{\pi/2} + \frac{m-1}{m+n} \int_0^{\pi/2} \sin^{m-2} x \cos^n x dx$

P-T.O. (9)

$$\int_0^{\pi/2} \sin^m x \cos^n x dx = \frac{m-1}{m+n} \int_0^{\pi/2} \sin^{m-2} x \cos^n x dx$$

let $I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x dx$, Then

$$I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}$$

Replacing m by $m-2, m-4, \dots, 3, 2$, we obtain

$$I_{m-2,n} = \frac{m-3}{m+n-2} I_{m-4,n}$$

$$I_{m-4,n} = \frac{m-5}{m+n-4} I_{m-6,n}$$

.....

$$m=3, \quad I_{3,n} = \frac{2}{3+n} I_{1,n} \quad \text{if } m \text{ is odd}$$

$$m=2, \quad I_{2,n} = \frac{1}{2+n} I_{0,n} \quad \text{if } m \text{ is even}$$

$$\therefore I_{m,n} = \begin{cases} \frac{m-1}{m+n} \cdot \frac{m-3}{m+n-2} \cdot \frac{m-5}{m+n-4} \cdots \frac{2}{3+n} I_{1,n} & \text{if } m \text{ is odd.} \\ \frac{m-1}{m+n} \cdot \frac{m-3}{m+n-2} \cdot \frac{m-5}{m+n-4} \cdots \frac{1}{2+n} I_{0,n} & \text{if } m \text{ is even.} \end{cases}$$

$$I_{1,n} = \int_0^{\pi/2} \sin x \cos^n x dx = - \left[\frac{\cos^{n+1} x}{n+1} \right]_0^{\pi/2} = \frac{1}{n+1}$$

$$\text{and } I_{0,n} = \int_0^{\pi/2} \cos^n x dx = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3} & \text{if } n \text{ is odd} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2} & \text{if } n \text{ is even} \end{cases}$$

ctd...

∴ For m odd

$$I_{m,n} = \frac{m-1}{m+n} \cdot \frac{m-3}{m+n-2} \cdots \frac{2}{3+n} \cdot \frac{1}{1+n}, \quad n \text{ may be even or odd}$$

For m even

$$I_{m,n} = \begin{cases} \frac{m-1}{m+n} \cdot \frac{m-3}{m+n-2} \cdots \frac{1}{2+n} \cdot \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3}; & n \text{ is odd} \\ \frac{m-1}{m+n} \cdot \frac{m-3}{m+n-2} \cdots \frac{1}{2+n} \cdot \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2}; & n \text{ is even} \end{cases}$$

□

Question. If $I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x dx,$

Prove that $I_{m,n} = \frac{n-1}{m+n} I_{m,n-2};$

where m and n are positive integers.

□