

## Infinite Limits:

The expressions  $\lim_{x \rightarrow a^-} f(x) = +\infty$  and  $\lim_{x \rightarrow a^+} f(x) = +\infty$

denote that  $f(x)$  increases without bound as  $x$  approaches  $a$  from the left and from the right respectively. If both are true, then we write

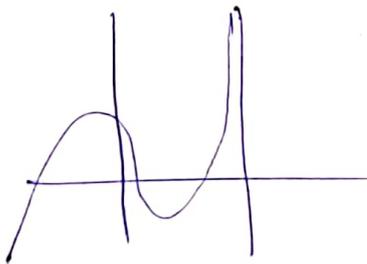
$$\lim_{x \rightarrow a} f(x) = +\infty.$$

Similarly, the expressions  $\lim_{x \rightarrow a^-} f(x) = -\infty$  and  $\lim_{x \rightarrow a^+} f(x) = -\infty$

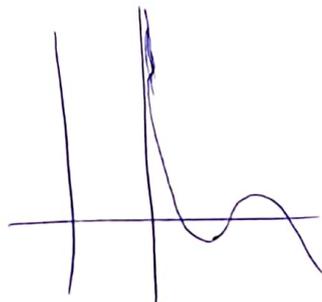
denote that  $f(x)$  decreases without bound as  $x$  approaches  $a$  from the left and from the right, respectively. If both are true, then we write

$$\lim_{x \rightarrow a} f(x) = -\infty.$$

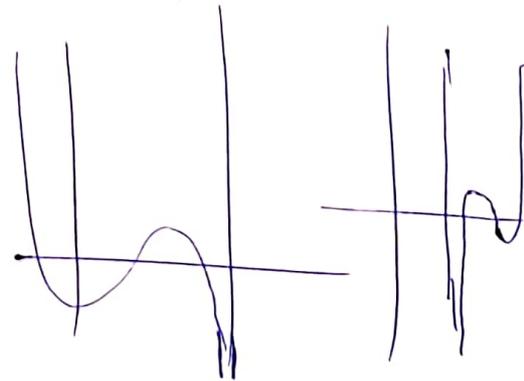
## Vertical Asymptotes:



$$\lim_{x \rightarrow a^-} f(x) = +\infty$$



$$\lim_{x \rightarrow a^+} f(x) = +\infty$$



$$\lim_{x \rightarrow a^-} f(x) = -\infty$$

$$\lim_{x \rightarrow a^+} f(x) = -\infty$$

In each case the graph of  $y = f(x)$  either rises or falls without bound, squeezing closer and closer to the vertical line  $x = a$  as  $x$  approaches  $a$  from the side indicated in the limit. The line  $x = a$  is called a vertical asymptote of the curve  $y = f(x)$ .

## Limit at Infinity and Horizontal Asymptotes:

If the value of  $f(x)$  eventually get as close as we like to a number  $L$  as  $x$  increases without bound, then we write

$$\lim_{x \rightarrow +\infty} f(x) = L \quad \text{or} \quad f(x) \rightarrow L \text{ as } x \rightarrow +\infty.$$

Similarly, if the value of  $f(x)$  eventually get as close as we like to a number  $L$  as  $x$  decreases without bound, then we write

$$\lim_{x \rightarrow -\infty} f(x) = L \quad \text{or} \quad f(x) \rightarrow L \text{ as } x \rightarrow -\infty$$

If either limit holds, then we call the line  $y = L$  a horizontal asymptotes for the graph of  $f$ .

## Infinite Limits at Infinity:

If the value of  $f(x)$  increases without bound as  $x \rightarrow +\infty$  or as  $x \rightarrow -\infty$ , Then we write

$$\lim_{x \rightarrow +\infty} f(x) = +\infty \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = +\infty.$$

and if the value of  $f(x)$  decrease without bound as  $x \rightarrow +\infty$  or as  $x \rightarrow -\infty$ , then we write

$$\lim_{x \rightarrow +\infty} f(x) = -\infty \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = -\infty.$$

## Limit at infinity :

Let  $f(x)$  be defined for all  $x$  in some infinite open interval extending in the positive  $x$ -direction. We write

$\lim_{x \rightarrow +\infty} f(x) = L$  if given any number  $\varepsilon > 0$ , there

corresponds a positive number  $N$  such that

$$\boxed{|f(x) - L| < \varepsilon \quad \text{if } x > N.}$$

Similarly, let  $f(x)$  be defined for all  $x$  in some infinite open interval extending in the negative  $x$ -direction.

We write  $\lim_{x \rightarrow -\infty} f(x) = L$  if given any number  $\varepsilon > 0$ ,

there corresponds a negative number  $N$  such that

$$\boxed{|f(x) - L| < \varepsilon \quad \text{if } x < N}$$

Eg. Prove that  $\lim_{x \rightarrow +\infty} \frac{1}{x} = 0$

Sol<sup>n</sup>. With  $f(x) = \frac{1}{x}$  and  $L = 0$ , we must show that given  $\varepsilon > 0$ , we can find a number  $N > 0$  such that

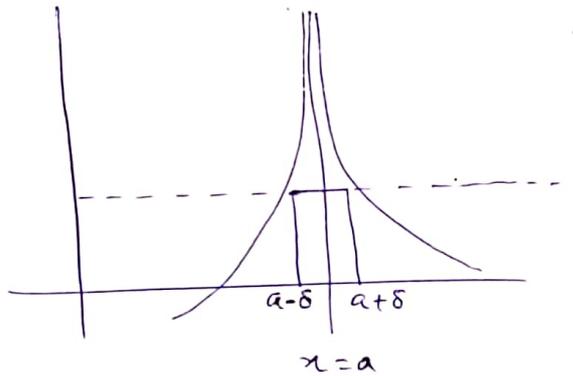
$$\left| \frac{1}{x} - 0 \right| < \varepsilon \quad \text{if } x > N$$

$$\text{or } \frac{1}{x} < \varepsilon \quad \text{if } x > N \quad \left\{ \because x \rightarrow +\infty \Rightarrow x > 0 \right\}$$

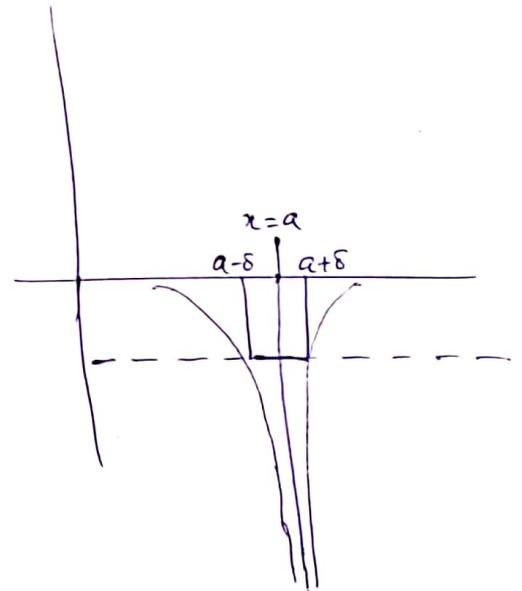
$$\text{or } x > \frac{1}{\varepsilon} \quad \text{if } x > N$$

$$\therefore N = \frac{1}{\varepsilon} \quad \text{and we have } \lim_{x \rightarrow +\infty} \frac{1}{x} = 0$$

## Infinite Limits:



$$f(x) > M \text{ if } 0 < |x-a| < \delta$$



$$f(x) < M \text{ if } 0 < |x-a| < \delta$$

Let  $f(x)$  be defined for all  $x$  in some open interval containing  $a$ , except that  $f(x)$  need not be defined at  $a$ .

Then we write  $\lim_{x \rightarrow a} f(x) = +\infty$  if given any positive

number  $M$ , we can find a number  $\delta > 0$  such that

$$f(x) > M \text{ if } 0 < |x-a| < \delta.$$

Similarly, Let  $f(x)$  be defined for all  $x$  in some open interval containing  $a$ , except that  $f(x)$  need not be defined at  $a$ . We write

$$\lim_{x \rightarrow a} f(x) = -\infty$$

if given any negative number  $M$ , we can find a number  $\delta > 0$  such that

$$f(x) < M \text{ if } 0 < |x-a| < \delta.$$

Example: Prove that  $\lim_{x \rightarrow 0} \frac{1}{x^2} = +\infty$

Solution: With  $f(x) = \frac{1}{x^2}$  and  $a=0$ ,

we must show that given a number  $M > 0$ ,  
we can find a number  $\delta > 0$  such that

$$\frac{1}{x^2} > M \quad \text{if} \quad 0 < |x-0| < \delta$$

$$\text{or} \quad x^2 < \frac{1}{M} \quad \text{if} \quad 0 < |x| < \delta$$

$$\text{or} \quad |x| < \frac{1}{\sqrt{M}} = \delta \quad \text{if} \quad 0 < |x| < \delta$$

and we have

$$\lim_{x \rightarrow 0} \frac{1}{x^2} = +\infty.$$

Example: Find  $\lim_{x \rightarrow +\infty} \frac{5x^3 - 2x^2 + 1}{1 - 3x}$ .

Solution.

$$\lim_{x \rightarrow +\infty} \frac{5x^3 - 2x^2 + 1}{1 - 3x} = \lim_{x \rightarrow +\infty} \frac{x(5x^2 - 2x + \frac{1}{x})}{x(\frac{1}{x} - 3)}$$

$$= \frac{\lim_{x \rightarrow +\infty} (5x^2 - 2x) + \lim_{x \rightarrow +\infty} \frac{1}{x}}{\lim_{x \rightarrow +\infty} \frac{1}{x} - \lim_{x \rightarrow +\infty} 3}$$

$$= \frac{\infty + 0}{0 - 3} = -\infty$$