Open and Closed Sets

Definition 1.

Let A be a subset of a metric space (X, d). A point $x \in X$ is called an **interior point** of A if there exists an open ball with centre x contained in A, i.e.,

$$x \in S(x, r) \subseteq A \text{ for some } r > 0,$$

or equivalently, if x has a neighbourhood contained in A.

The set of all interior points of A is called the **interior of** A and is denoted either by Int(A) or A°.

$$\text{Int}(A) = A^{\circ} = \{x \in A: S(x, r) \subseteq A \text{ for some } r > 0\}.$$

Observe that Int(A) $\subseteq A$.

Result 1.

Let A be a subset of a metric space (X, d). Then

(i) A° is an open subset of A that contains every open subset of A;
(ii) A is open if and only if $A = A^{\circ}$.

Proof. (i) Let $x \in A^\circ$ be arbitrary. Then, by definition, there exists an open ball $S(x, r) \subseteq A$. But $S(x,r)$ being an open set (see Theorem 2.1.5), each point of it is the centre of some open ball contained in $S(x,r)$ and consequently also contained in A. Therefore each point of $S(x,r)$ is an interior point of A, i.e., $S(x, r) \subseteq A^\circ$. Thus, x is the centre of an open ball contained in A°. Since $x \in A^\circ$ is arbitrary, it follows that each $x \in A^\circ$ has the property of being the centre of an open ball contained in A°. Hence, A° is open.

It remains to show that A° contains every open subset $G \subseteq A$. Let $x \in G$. Since G is open, there exists an open ball $S(x, r) \subseteq G \subseteq A$. So $x \in A^\circ$. This shows that $x \in G \Rightarrow x \in A^\circ$. In other words, $G \subseteq A^\circ$.

(ii) is immediate from (i). □

Result 2.

Let (X, d) be a metric space and A, B be subsets of X. Then

(i) $A \subseteq B \Rightarrow A^\circ \subseteq B^\circ$;
(ii) $(A \cap B)^\circ = A^\circ \cap B^\circ$;
(iii) $(A \cup B)^\circ \supseteq A^\circ \cup B^\circ$.

Proof. (i) Let $x \in A^\circ$. Then there exists an $r > 0$ such that $S(x, r) \subseteq A$. Since $A \subseteq B$, we have $S(x, r) \subseteq B$, i.e., $x \in B^\circ$.

(ii) $A \cap B \subseteq A$ as well as $A \cap B \subseteq B$. It follows from (i) that $(A \cap B)^\circ \subseteq A^\circ$ as well as $(A \cap B)^\circ \subseteq B^\circ$, which implies that $(A \cap B)^\circ \subseteq A^\circ \cap B^\circ$. On the other hand, let $x \in A^\circ \cap B^\circ$. Then $x \in A^\circ$ and $x \in B^\circ$. Therefore, there exist $r_1 > 0$ and $r_2 > 0$ such that $S(x, r_1) \subseteq A$ and $S(x, r_2) \subseteq B$. Let $r = \min\{r_1, r_2\}$. Clearly, $r > 0$ and $S(x, r) \subseteq A \cap B$, i.e., $x \in (A \cap B)^\circ$.

(iii) $A \subseteq A \cup B$ as well as $B \subseteq A \cup B$. Now apply (i). □

Note.

The following example shows that $(A \cup B)^\circ$ need not be the same as $A^\circ \cup B^\circ$. Indeed, if $A = [0,1]$ and $B = [1,2]$, then $A \cup B = [0,2]$. Since $A^\circ = (0,1), B^\circ = (1,2)$ and $(A \cup B)^\circ = (0,2)$, we have $(A \cup B)^\circ \neq A^\circ \cup B^\circ$.

Definition 2.

Let X be a metric space and F a subset of X. A point $x \in X$ is called a **limit point** of F if each open ball with centre x contains at least one point of F different from x, i.e.,

$$(S(x,r) - \{x\}) \cap F \neq \emptyset.$$

The set of all limit points of F is denoted by F' and is called the **derived set** of F.

Example1.

(i) The subset $F = \{1, 1/2, 1/3, \ldots\}$ of the real line has 0 as a limit point; in fact, 0 is its only limit point. Thus the derived set of F is $\{0\}$, i.e., $F' = \{0\}$.

(ii) The subset $\mathbb{Z}$ of integers of the real line, consisting of all the integers, has no limit point. Its derived set $\mathbb{Z}'$ is $\emptyset$.

(iii) Each real number is a limit point of the subset of rationals: $\mathbb{Q}' = \mathbb{R}$.

(iv) If (X, d) is a discrete metric space and $F \subseteq X$, then F has no limit points, since every open ball of radius 1 consists only of the centre.

Result 3.

Let (X, d) be a metric space and $F \subseteq X$. If x_0 is a limit point of F, then every open ball $S(x_0, r)$, $r > 0$, contains an infinite number of points of F.

Proof. Suppose that the ball $S(x_0, r)$ contains only a finite number of points of F. Let $y_1, y_2, \ldots, y_n$ denote the points of $S(x_0, r) \cap F$ that are distinct from x_0. Let

$$\delta = \min\{d(y_1, x_0),\ d(y_2, x_0), \ldots, d(y_n, x_0)\}.$$

Then the ball $S(x_0, \delta)$ contains no point of F distinct from x_0, contradicting the assumption that x_0 is a limit point of F. □

Result 4.

Let (X, d) be a metric space and $F \subseteq X$. Then a point x_0 is a limit point of F if and only if it is possible to select from the set F a sequence of distinct points $x_1, x_2, \ldots, x_n, \ldots$ such that $\lim_n d(x_n, x_0) = 0$.

Proof. If $\lim_n d(x_n, x_0) = 0$, where $x_1, x_2, \ldots, x_n, \ldots$ is a sequence of distinct points of F, then every ball $S(x_0, r)$ with centre x_0 and radius r contains each of x_n, where $n \geq n_0$ for some suitably chosen n_0. As $x_1, x_2, \ldots, x_n, \ldots$ in F are distinct, it follows that $S(x_0, r)$ contains a point of F different from x_0. So, x_0 is a limit point of F.

On the other hand, assume that x_0 is a limit point of F. Choose a point $x_1 \in F$ in the open ball $S(x_0, 1)$ such that x_1 is different from x_0. Next, choose a point $x_2 \in F$ in the open ball $S(x_0, 1/2)$ different from x_0 as well as from x_1; this is possible by Proposition 2.1.19. Continuing this process in which, at the nth step of the process we choose a point $x_n \in F$ in $S(x_0, 1/n)$ different from $x_1, x_2, \ldots, x_{n-1}$, we have a sequence $\{x_n\}$ of distinct points of the set F such that $\lim_n d(x_n, x_0) = 0$. □

Definition 3.

A subset F of the metric space (X, d) is said to be **closed** if it contains each of its limit points, i.e., $F' \subseteq F$.

Example.

(i) The set **Z** of integers is a closed subset of the real line.

(ii) The set $F = \{1, 1/2, 1/3, \ldots, 1/n, \ldots\}$ is not closed in **R**. In fact, $F' = \{0\}$, which is not contained in F.

(iii) Each subset of a discrete metric space is closed.

Result 5.

Let F be a subset of the metric space (X, d). The set of limit points of F, namely, F' is a closed subset of (X, d), i.e., $(F')' \subseteq F'$.

Proof. If $F' = \emptyset$ or $(F')' = \emptyset$, then there is nothing to prove. Let $F' \neq \emptyset$ and let $x_0 \in (F')'$. Choose an arbitrary open ball $S(x_0, r)$ with centre x_0 and radius r. By the definition of limit point, there exists a point $y \in F'$ such that $y \in S(x_0, r)$. If $r' = r - d(y, x_0)$, then $S(y, r')$ contains infinitely many points of F by Proposition 2.1.19. But $S(y, r') \subseteq S(x_0, r)$ as in the proof of Theorem 2.1.5. So, infinitely many points of F lie in $S(x_0, r)$. Therefore, x_0 is a limit point of F, i.e., $x_0 \in F'$. Thus, F' contains all its limit points and hence F' is closed. □

Result 6.

Let (X, d) be a metric space and let F_1, F_2 be subsets of X.

(i) If $F_1 \subseteq F_2$, then $F_1' \subseteq F_2'$.
(ii) $(F_1 \cup F_2)' = F_1' \cup F_2'$.
(iii) $(F_1 \cap F_2)' \subseteq F_1' \cap F_2'$.

Proof. The proofs of (i) and (iii) are obvious. For the proof of (ii), observe that $F_1' \cup F_2' \subseteq (F_1 \cup F_2)'$, which follows from (i). It remains to show that

$$(F_1 \cup F_2)' \subseteq F_1' \cup F_2'.$$

Let $x_0 \in (F_1 \cup F_2)'$. Then there exists a sequence $\{x_n\}_{n \geq 1}$ of distinct points in $F_1 \cup F_2$ such that $d(x_n, x_0) \to 0$ as $n \to \infty$.

If an infinite number of points x_n lie in F_1, then $x_0 \in F_1'$, and, consequently, $x_0 \in F_1' \cup F_2'$. If only a finite number of points of $\{x_n\}_{n \geq 1}$ lie in F_1, then $x_0 \in F_2' \subseteq F_1' \cup F_2'$. We therefore have $x_0 \in F_1' \cup F_2'$ in either case. This completes the proof of (ii). □

Definition 4.

Let F be a subset of a metric space (X, d). The set $F \cup F'$ is called the closure of F and is denoted by $\bar{F}$.

Thus,

The closure $\bar{F}$ of $F \subseteq X$, where (X, d) is a metric space, is closed.

Proof. In fact,

$$(\bar{F})' = (F \cup F')' = F' \cup (F')' \subseteq F' \cup F' = F' \subseteq \bar{F}.$$

Result 7.

(i) Let F be a subset of a metric space(X, d). Then F is closed if and only if $F = \bar{F}$.

(ii) If $A \subseteq B$, then $\bar{A} \subseteq \bar{B}$.

(iii) If $A \subseteq F$ and F is closed, then $\bar{A} \subseteq F$.

Proof. (i) If $F = \bar{F}$, then it follows that F is closed. On the other hand, suppose that F is closed; then

$$\bar{F} = F \cup F' = F \subseteq \bar{F}.$$

It follows from the above relations that $F = \bar{F}$.

Proof of (ii) and (iii) follows from previous results

Result 8.

Let (X, d) be a metric space and $F \subseteq X$. Then the following statements are equivalent:

(i) $x \in \bar{F}$;
(ii) $S(x,\varepsilon) \cap F \neq \emptyset$ for every open ball $S(x,\varepsilon)$ centred at x;
(iii) there exists an infinite sequence $\{x_n\}$ of points (not necessarily distinct) of F such that $x_n \to x$.

Proof. (i)⇒(ii). Let $x \in \bar{F}$. If $x \in F$, then obviously $S(x,\varepsilon) \cap F \neq \emptyset$. If $x \notin F$, then by the definition of closure, we have $x \in F'$. By definition of a limit point,

$$(S(x,\ \varepsilon)\backslash\{x\}) \cap F \neq \emptyset$$

and, a fortiori,

$$S(x,\ \varepsilon) \cap F \neq \emptyset.$$

(ii)⇒(iii). For each positive integer n, choose $x_n \in S(x, 1/n) \cap F$. Then the sequence $\{x_n\}$ of points in F converges to x. In fact, upon choosing $n_0 > 1/\varepsilon$,

where $\varepsilon > 0$ is arbitrary, we have $d(x_n, x) < 1/n < 1/n_0 < \varepsilon$, i.e., $x_n \in S(x,\ \varepsilon)$ whenever $n \geq n_0$.

(iii)⇒(i) If the sequence $\{x_n\}_{n\geq 1}$ of points in F consists of finitely many distinct points, then there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} = x$ for all k. So, $x \in F$. If however, $\{x_n\}_{n\geq 1}$ contains infinitely many distinct points, then there exists a subsequence $\{x_{n_k}\}$ consisting of distinct points and $\lim_k d(x_{n_k},\ x) = 0$, for

$\lim_n d(x_n,\ x) = 0$ by hypothesis.

it follows that $x \in F' \subseteq \bar{F}$.

Result 9.

Let $F_1,\ F_2$ be subsets of a metric space (X, d). Then

(i) $\overline{(F_1 \cup F_2)} = \bar{F}_1 \cup \bar{F}_2$;
(ii) $\overline{(F_1 \cap F_2)} \subseteq \bar{F}_1 \cap \bar{F}_2$.

Proof. We have

$$\overline{(F_1 \cup F_2)} = (F_1 \cup F_2) \cup (F_1 \cup F_2)' = (F_1 \cup F_2) \cup (F_1' \cup F_2')$$
$$= (F_1 \cup F_1') \cup (F_2 \cup F_2') = \bar{F}_1 \cup \bar{F}_2,$$

which establishes (i). The proof of (ii) is equally simple.

Result 10.

Let (X, d) be a metric space. The empty set $\varnothing$ and the whole space X are closed sets.

Proof. Since the empty set has no limit points, the requirement that a closed set contain all its limit points is automatically satisfied by the empty set.

Since the whole space contains all points, it certainly contains all its limit points (if any), and is thus closed. □

Result 11.

Let (X, d) be a metric space and F be a subset of X. Then F is closed in X if and only if F^c is open in X.

Proof. Suppose F is closed in X. We show that F^c is open in X. If $F = \emptyset$(respectively, X), then $F^c = X$ (respectively, $\emptyset$) and it is open by Theorem 2.1.7(i); so we may suppose that $F \neq \emptyset \neq F^c$. Let x be a point in F^c. Since F is closed and $x \notin F$, x cannot be a limit point of F. So there exists an $r > 0$ such that $S(x, r) \subseteq F^c$. Thus, each point of F^c is contained in an open ball contained in F^c. This means F^c is open.

For the converse, suppose F^c is open. We show that F is closed. Let $x \in X$ be a limit point of F. Suppose, if possible, that $x \notin F$. Then $x \in F^c$, which is assumed to be open. Therefore, there exists $r > 0$ such that $S(x, r) \subseteq F^c$, i.e.,

$$S(x, r) \cap F = \emptyset.$$

Thus, x cannot be a limit point of F, which is a contradiction. Hence, x belongs to F. □

Result 12.

Let (X, d) be a metric space and $\bar{S}(x, r) = \{y \in X: d(y, x) \leq r\}$ be a closed ball in X. Then $\bar{S}(x, r)$ is closed.

Proof. We show that $(\bar{S}(x,r))^c$ is open in X (see Theorem 2.1.32). Let $y \in (\bar{S}(x,r))^c$. Then $d(y,x) > r$. If $r_1 = d(y,x) - r$, then $r_1 > 0$. Moreover, $S(y,r_1) \subseteq (\bar{S}(x,r))^c$. Indeed, if $z \in S(y,r_1)$, then

$$d(z,x) \geq d(y,x) - d(y,z) > d(y,x) - r_1 = r.$$

Thus, $z \notin \bar{S}(x,r)$, i.e., $z \in (\bar{S}(x,r))^c$. □

Result 13.

Let (X,d) be a metric space. Then

(i) $\varnothing$ and X are closed;

(ii) any intersection of closed sets is closed;

(iii) a finite union of closed sets is closed.

Proof. (i) This follows from above result 10.

(ii) Let $\{F_\alpha\}$ be a family of closed sets in X and $F = \bigcap_\alpha F_\alpha$. Then F is closed if F^c is open. Since $F^c = \bigcup_\alpha F_\alpha^c$ by de Morgan's laws, and since each F_α^c is open , $\bigcup_\alpha F_\alpha^c$ is open , by previous result,

i.e., F^c is open.

(iii) This proof is similar to (ii).

Note.

An arbitrary union of closed sets need not be closed. Indeed, $\bar{S}(0, 1-1/n)$, $n \geq 2$, is a closed subset of the complex plane, but

$$\bigcup_{n=2}^{\infty} \bar{S}\left(0, 1-\frac{1}{n}\right) = S(0,1)$$

is not closed (because each point z satisfying $|z| = 1$ is a limit point of $S(0,1)$ but is not contained in $S(0,1)$).